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Functional Integrals in the Statistical Theory of Turbulence and the Burgers Model Equation

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The functional integrals appearing in master equations for turbulent flow of an incompressible fluid and for the Burgers model equation are treated. A possible way is described to define the integration properly and related problems are discussed. For the simple example of the Burgers model equation of turbulence some results are presented.

KEY WORDS: Projection operators; functional integrals; turbulence; Burgers equation.

1. INTRODUCTION

The theoretical treatment of turbulence has paid much attention so far to perturbation methods about laminar flow,[(1)] turbulent flow,[(2)] and Gaussian random processes[(3)] by using the moment formulation, and for the distribution function formalism this also has been done about the same states for

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laminar flow,[5] turbulent flow,[6] Gaussian random processes.[6] Lee[7] gives an excellent survey and introduces the general kinetic equation well known from nonequilibrium statistical mechanics. This kinetic equation is closely related to Zwanzig's master equation.[11] Its derivation involves diagrammatic techniques developed by the Brussels school,[8] whereas Zwanzig uses projection operators. In this paper the explicit calculation of functional integrals appearing in master equations for turbulent flow and especially for the Burgers model for large Reynolds numbers is investigated. Conditions for the proper definition of this integration process are considered and for the Burgers model some integrals are explicitly calculated.

2. BASIC EQUATIONS

The general formulation of the turbulence problem for incompressible Newtonian fluids leads to Hopf's equation[9] for the characteristic functional $\theta[\varphi_\alpha(\cdot)]$ of the probability measure,

$$\frac{\partial \theta}{\partial t} = \int_R \tilde{\varphi}_\alpha(\xi)\left[i\frac{\partial}{\partial \xi_\beta}\frac{\delta^2\theta}{\delta\varphi_\beta(\xi)\,\delta\varphi_\alpha(\xi)\,d\xi\,d\xi} + \nu\Delta_\xi\frac{\delta\theta}{\delta\varphi_\alpha(\xi)\,d\xi}\right]d\xi \tag{1}$$

R denotes the domain of the φ_α with sufficiently smooth boundary ∂R and $\tilde{\varphi}_\alpha$ is the solenoidal part of φ_α. The alternative is the (generalized) Liouville equation for the probability density functional F

$$\frac{\partial F}{\partial t} + \int_R d\xi\,\frac{\delta}{\delta\varphi_\alpha(\xi)\,d\xi}\,[Q_\alpha(\varphi_\beta)\cdot F[\varphi_\alpha(\cdot),t]] = 0 \tag{2}$$

where $\delta/\delta\varphi_\alpha(\xi)\,d\xi$ denotes the Frechet derivative and $\partial\varphi_\alpha/\partial t = Q_\alpha(\varphi_\beta)$ is the rate of change of the argument φ_α of F. These two equations are equivalent, as was shown by Keller.[10] The probability functional F is defined on the phase space Ω of all possible realizations of the fluctuating flow field. We are now interested in the application of an idea by Zwanzig,[11] who used projection operators to obtain equations for the probability density of the values of rather arbitrary functionals (scalar- or vector-valued) defined on the phase space Ω. Thus the closure problem for pdf's can be avoided under certain initial conditions imposed on F. Start from (2) written in the form

$$\partial F/\partial t = i\mathscr{L}F \tag{3}$$

where $\mathscr{L}$ denotes the Liouville operator

$$i\mathscr{L}\circ \equiv \int_R d\xi\,\frac{\delta}{\delta\varphi_\alpha(\xi)\,d\xi}\,[Q_\alpha(\varphi_\beta)\circ] \tag{4}$$

The domain of definition of $i\mathscr{L}$ is the class of functionals which are defined on Ω and are weakly differentiable. We note that $i\mathscr{L}$ is a linear operator. For the choice of the phase function $A[\varphi_\alpha(\cdot)]$ we suggest

$$A[\varphi_\alpha(\cdot)] = \varphi_\alpha(\mathbf{x}_0) \tag{5}$$

Since in this form A is not regular, we consider the fundamental sequence

$$A_n[\varphi_\alpha(\cdot)] = \frac{1}{V_n}\int_R d\xi\, \chi_n(\xi)\varphi_\alpha(\xi), \qquad V_n \to 0 \quad \text{as } n \to \infty$$

where χ_n is the characteristic function of V_n centered around $\mathbf{x}_0$. The A_n are now regular functionals. In the limit $n \to \infty$, A_n therefore gives the velocity at a chosen point $\mathbf{x}_0 \in R$. Following Ref. 11, we can derive an equation for the values u_α of A by introducing the projection operator

$$P_A \circ \equiv \frac{1}{W(A[\varphi_\alpha])}\int_\Omega \mu(d\varphi')\, \delta(A[\varphi_\alpha'] - A[\varphi_\alpha]) \circ \tag{6}$$

where

$$W(A[\varphi_\alpha]) \equiv \int_\Omega \mu(d\varphi')\, \delta(A[\varphi_\alpha'] - A[\varphi_\alpha]) \tag{7}$$

Definition (6) includes the choice of a proper measure μ on $(\Omega, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra of subsets of Ω. The construction of μ and related problems will be discussed below. The result of Zwanzig's derivation can be stated in the form of a linear integrodifferential equation for the pdf $g(u_\alpha, t)$. First $F[\varphi_\alpha(\cdot), t]$ is split into orthogonal parts

$$F = P_A F + (I - P_A)F$$

and we write $f_1 = P_A F$ and $f_2 = (I - P_A)F$. Then the pdf $g(u_\alpha, t)$ is related to f_1 by

$$g(u_\alpha, t) = \int_\Omega \mu(d\varphi)\, \delta(A[\varphi_\alpha] - u_\alpha)F[\varphi_\alpha(\cdot), t] \tag{8}$$

Furthermore, we define the averaging procedure with respect to the measure μ under the condition that $A[\varphi_\alpha(\cdot)]$ assumes a given value u_α as

$$\langle G[\varphi_\alpha(\cdot)], u_\alpha\rangle \equiv \frac{1}{W(u_\alpha)}\int_\Omega \mu(d\varphi)\, \delta(A[\varphi_\alpha] - u_\alpha)G[\varphi_\alpha] \tag{9}$$

where G is any regular functional. According to Ref. 11, the equation for $g(u_\alpha, t)$ can finally be given in the form

$$\begin{aligned} \frac{i}{W(u_\alpha)}\cdot\frac{\partial g}{\partial t} &= \langle \mathscr{L}f_1; u_\alpha\rangle + \langle \mathscr{L}\{\exp[-t(I - P_A)i\mathscr{L}]\}f_2(\varphi_\alpha, 0); u_\alpha\rangle \\ &\quad - i\int_0^t ds\, \langle i\mathscr{L}\{\exp[-s(I - P_A)i\mathscr{L}]\}(I - P_A)i\mathscr{L}f_1(\varphi_\alpha, t - s); u_\alpha\rangle \end{aligned} \tag{10}$$

Note that for $f_2[\varphi_\alpha, 0] = 0$ this is a closed and exact equation for $f_1[\varphi_\alpha\, t]$, because g and f_1 are related by

$$f_1[\varphi_\alpha(\cdot), t] = \int_{-\infty}^{\infty} du\ \delta(A_n[\varphi_\alpha(\cdot)] - u_\alpha)\frac{g(u_\alpha, t)}{W(u_\alpha)} \tag{11}$$

The condition $f_2[\varphi_\alpha, 0] = 0$ means that initially $F[\varphi_\alpha, 0] = f_1[\varphi_\alpha, 0]$ or in physical terms, for the phase functional chosen here, at $t = 0.0$ the fluctuations are restricted to the volume V_n. For the example of the Burgers model equation we shall give later a more refined version of (10). The change in time of a point φ_α in Ω is determined by the Navier–Stokes equations and the continuity equation for the case of turbulence. For the Burgers model this equation is given by

$$\frac{\partial\varphi}{\partial t} = -\varphi\frac{\partial\varphi}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2\varphi}{\partial x^2} \tag{11a}$$

where $\varphi(x)$ is defined on the x interval $[-T, T]$, $T > 0$, and is subjected to the boundary conditions

$$\varphi(T) = \varphi(-T) = 0$$

3. THE STRUCTURE OF THE PHASE SPACE Ω

Consider first the properties of a possible realization of the fluctuating flow field $\varphi_\alpha(\mathbf{x})$. The function $\varphi_\alpha(\mathbf{x})$ must be twice differentiable

$$\frac{\partial^2}{\partial x_\beta\,\partial x_\gamma}\varphi_\alpha(\mathbf{x}) \in L_R{}^2; \qquad \mathbf{x} \in R$$

and it must satisfy the boundary conditions

$$\varphi_\alpha(\mathbf{x}) = 0; \qquad \mathbf{x} \in \partial R$$

or in the case of homogeneous turbulence this is replaced by the condition of finite norm and finally the continuity equation

$$\frac{\partial}{\partial x_\alpha}\varphi_\alpha(\mathbf{x}) = 0; \qquad \mathbf{x} \in R$$

must be fulfilled. From these properties it follows that Ω must be a linear subspace of the special Sobolev space

$$W_2^{(2)}(R) = \left\{\varphi_\alpha(\mathbf{x}):\ \ \frac{\partial^2}{\partial x_\beta\,\partial x_\gamma}\varphi_\alpha \in L_R{}^2,\ \ \varphi_\alpha(\mathbf{x}) = 0 \quad \forall\mathbf{x} \in \partial R\right\}$$

with the norm

$$\|\varphi_\alpha\| = \left\{\sum_{l=0}^{2}\int_R d\xi\ D^l\varphi_\alpha \cdot D^l\varphi_\alpha\right\}^{1/2}$$

where

$$D^l \cdot \equiv \frac{\partial^l \cdot}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}}, \qquad l = l_1 + l_2 + l_3$$

In this special case $W_2^{(2)}(R)$ is a separable Hilbert space[12] with scalar product given by

$$(\varphi_\alpha, \psi_\alpha) = \sum_{l=0}^{2} \int_R d\xi \; D^l\varphi_\alpha \cdot D^l\psi_\alpha \tag{12}$$

Then by (12) the structure of Ω as a subspace of $W_2^{(2)}(r)$ is uniquely determined. Since $W_2^{(2)}(r)$ is a separable Hilbert space, there exists an orthonormalized basis $\{\mathbf{e}^k\}_{k=1}^{\infty}$ in $W_2^{(2)}(R)$. Consequently, we can construct a basis $\{\mathbf{f}^k\}$ using a continuous linear mapping Φ: $W_2^{(2)}(R) \to \Omega$. The basis turns out to be very useful in constructing a Gaussian measure μ over $W_2^{(2)}(R)$ and Ω, respectively.

For the Burgers model the phase space has a much simpler structure. It is defined by

$$\Omega = \left\{\varphi(x): \; \frac{d^2\varphi}{dx^2} \in L_R^2, \quad \varphi(T) = \varphi(-T) = 0\right\}$$

and we note that $\Omega = W_2^{(2)}[-T, T]$.

4. INTEGRATION OF FUNCTIONALS OVER Ω

Consider a measure μ on $(W_2^{(2)}(R), \mathscr{B})$, where $\mathscr{B}$ is the Borel algebra and a sequence of projections $\{P_n\}$ of $W_2^{(2)}(R)$ into finite-dimensional subspaces $L_n(R) \subset W_2^{(2)}(R)$ with the property

$$\lim_{n\to\infty} P_n W_2^{(2)}(R) = W_2^{(2)}(R)$$

The measure μ generates on each L_n the measures

$$\mu_n(A) = \mu(\{\varphi_\alpha : P_n\varphi_\alpha \in A\}), \qquad \forall A \in \mathscr{B}$$

Analogously, P_n creates cylinder functionals on L_n by $F_n(\varphi_\alpha) = F(P_n\varphi_\alpha)$, where F is a functional defined on $W_2^{(2)}(R)$. These measures are compatible (Ref. 13, §2). It would be very easy now to construct a measure μ on $(W_2^{(2)}(R), \mathscr{B})$ if to each compatible sequence $\{\mu_n\}$ of finite-dimensional distributions there corresponded a measure μ as $n \to \infty$. But this is not the case and the sequence $\{\mu_n\}$ has to satisfy an additional condition stated in Lemma 1 of Ref. 13, §2, in terms of μ_n or in the Minlos–Sazonow theorem (Ref. 13, §4) in terms of the characteristic functional $\theta(z_\alpha)$ generated by the sequence $\{\mu_n\}$. But the situation is not hopeless because we can define integration with respect to weak distributions that do not generate measures (see Ref. 13, §2), though we possibly sacrifice σ-addivity of the integral. Therefore we consider both options: integration with respect to a weak distribution and integration with respect to a measure. The reason for this is the fact that in certain cases the calculations for weak distributions are easier than for measures.

Now consider the special spaces in this context. To construct a measure on $(\Omega\,\mathscr{B})$ we proceed as follows. First we establish a linear mapping Φ of $W_2^{(2)}(R)$ onto Ω. A possible choice of Φ for the case $R = R^3$ (no boundaries) is defined as

$$\varphi_\alpha(\mathbf{x}) = \Phi\tilde{\varphi}_\alpha(\mathbf{x}), \qquad \forall\tilde{\varphi}_\alpha \in W_2^{(2)}(R)$$

$$\Phi\tilde{\varphi}_\alpha(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dk \int_R dy\,\{\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]\}\tilde{\varphi}_\beta(\mathbf{y})\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right]$$

for all $\tilde{\varphi}_\alpha \in W_2^{(2)}(R)$. A simple calculation shows that from $\tilde{\varphi}_\alpha \in \Omega$ it follows that $\Phi\tilde{\varphi}_\alpha = \tilde{\varphi}_\alpha$ and that $\partial\varphi_\alpha/\partial x_\alpha = 0$ is indeed satisfied and therefore $\Phi W_2^{(2)}(R) = \Omega$. In the case of a compact R with a smooth boundary ∂R and homogeneous boundary conditions we can resort to the linear mapping Φ' of $W_2^{(3)}R$ onto Ω defined by

$$\varphi_\alpha(\mathbf{x}) = \Phi'\tilde{\varphi}_\alpha(\mathbf{x}) = \nabla \times \tilde{\varphi}_\alpha$$

$$\tilde{\varphi}_\alpha(\mathbf{x}) \in W_2^{(3)}(R) = \{\tilde{\varphi}_\alpha(\mathbf{x}):\ \ D^3\tilde{\varphi}_\alpha \in L_R{}^2, \quad \nabla \times \tilde{\varphi}_\alpha = 0 \quad \forall_\beta \in \partial R\}$$

Then

$$\Phi'\tilde{\varphi}_\alpha \in \Omega$$

The condition $\partial\varphi_\alpha/\partial x_\alpha = 0$ is obviously satisfied and for $\varphi_\alpha \in \Omega$ there exists $\tilde{\varphi}_\alpha \in W_2^{(3)}(R)$ with $\varphi_\alpha = \Phi'\tilde{\varphi}_\alpha$ by means of Biot–Savart's well-known formula and therefore $\Phi' W_2^{(3)}(R) = \Omega$.

The second step is the definition of a measure μ on $(W_2{}^l(R), \mathscr{B})$, $l = 2, 3$. We shall restrict ourselves to Gaussian measures μ with zero mean for reasons discussed later. Then μ is uniquely defined (see Ref. 13, §5) by a nuclear operator $B_{\alpha\beta}$ and consequently the characteristic functional $\theta(z_\alpha)$ of μ has the form

$$\theta(z_\alpha) = \exp\{-\tfrac{1}{2}(B_{\alpha\beta}z_\alpha, z_\beta)\}, \qquad z_\alpha \in W_2^{(l)}\ (R)$$

$B_{\alpha\beta}$ nuclear means that $B_{\alpha\beta}$ is symmetric, nonnegative, and has finite trace. The construction of $B_{\alpha\beta}$ can be done by making use of the orthogonal and normalized basis $\{\mathbf{e}^k\}_{k=1}^{\infty}$ in $W_2^{(l)}(R)$. We define a class of operators $B_{\alpha\beta}$ by

$$B_{\alpha\beta} = \delta_{\alpha\beta}B_{\alpha\cdot} \quad \text{(no sum over } \alpha)$$

and the eigenvalues $\{\lambda_\alpha{}^k\}$ and eigenvectors $\{f_\alpha{}^k\}$ of B_α. If we restrict the eigenvalues to

$$\sum_{k=1}^{\infty} \lambda_\alpha{}^k < \infty, \qquad \alpha = 1, 2, 3$$

and $\lambda_\alpha^k > 0$, $k = 1, 2, \ldots$, $\alpha = 1, 2, 3$, and furthermore use the basis $\{\mathbf{e}^k\}_{k=1}^\infty$ as the corresponding set of eigenvectors, then it follows easily that

$$(B_{\alpha\beta}W_\alpha, z_\beta) = (W_\alpha, B_{\alpha\beta}z_\beta) \qquad \text{symmetry}$$
$$(B_{\alpha\beta}z_\alpha, z_\beta) \geqslant 0 \qquad \text{nonnegativity}$$

and

$$\operatorname{tr} B_{\alpha\beta} = \sum_{k=1}^{\infty} (B_{\alpha\beta}e_\alpha{}^k, e_\beta{}^k) \leqslant \sum_{k=1}^{\infty}\sum_{\alpha=1}^{3} \lambda_\alpha{}^k < \infty$$

as an easy calculation shows.

The final step is the construction of the measure μ_Ω on $(\Omega, \mathscr{B}_\Omega)$. To accomplish this we use the mapping Φ or Φ' constructed in the first step and define μ_Ω by

$$\mu_\Omega(C) = \mu(\Phi^{-1}C), \qquad C \in \mathscr{B}_\Omega$$

But in order to get an explicit expression for the density, $(d\mu/d\mu_\Omega)(\varphi_\alpha)$, according to Theorem 4 of Ref. 13, §25, has to be invertible, which is not the case. Therefore we map the basis $\{\mathbf{e}^k\}_{k=1}^\infty$ of $W_2^{(l)}(R)$ into Ω and construct μ_Ω directly by defining the nuclear operator $B_{\alpha\beta}$ in Ω using $\{\Phi\mathbf{e}^k\}$ orthonormalized if necessary.

For the integration with respect to a weak distribution generated by the sequence $\{\mu_n\}$ of finite-dimensional distributions

$$\mu_n(d\varphi) = \exp\{-\tfrac{1}{2}(P_n\varphi_\alpha, P_n\varphi_\alpha)\} \prod_{i=1}^{n} \left(\frac{\Delta_i}{2\pi}\right)^{3/2} d\varphi_i \tag{13}$$

we can employ the same procedure as described above to define the integral over Ω. For the Burgers model we note a special simplification. We denote by

$$[\varphi, \psi] = \int_R d\xi\, D^{(l)}\varphi \cdot D^{(l)}\psi \tag{14}$$

the scalar product $[\varphi, \psi]$ for $l = 2$. Then we can replace $(\cdot, \cdot)$ in (13) by $[\cdot, \cdot]$ and get a new sequence $\{\mu_n\}$ defining an integral (after appropriate normalization of the densities μ_n) more accessible to calculation.

Some remarks about the integration seem to be necessary at this point. Since the choice of μ in (6) is by no means unique, the solution of (10) will in general depend on μ. In order to make (6) a useful tool, we should satisfy the following condition on μ. The unknown probability measure F and μ must have the same mean for all times because in infinite-dimensional Hilbert spaces translationally invariant measures do not exist (see Ref. 13, §19). Therefore we have to introduce fluctuating fields with zero mean in the basic equations (1) and (2). Equation (8) tells us furthermore that the solution of (2) should be absolutely continuous with respect to μ. But this is an open question so far. In this context the Gaussian measure μ is a reasonable choice because according to Theorem 1, §26, in Ref. 13, there exists a large class of nonlinear transforms ν of a Gaussian measure μ that are equivalent to μ.

5. THE BURGERS MODEL

The phase space Ω allows in this example the construction of a 1–1 mapping Φ of $L_R{}^2$ onto Ω by

$$\varphi(x) \equiv \Phi y = \int_{-T}^{T} d\xi\, K(\xi, x) y(\xi), \qquad y \in L_R{}^2 \tag{15}$$

where

$$K(\xi, x) = \frac{1}{2}\,|\xi - x| - \frac{T}{2} + \frac{x\xi}{2T} \tag{16}$$

The projections P_n of Ω onto the finite-dimensional subspace Ω_n of n-step functions are given by

$$P_n\varphi = \{\varphi_i\}_{i=1}^{n}$$

where

$$\varphi_i = \frac{1}{|\xi_i - \xi_{i-1}|}\int_{\xi_{i-1}}^{\xi_i} d\xi\, \varphi(\xi), \qquad \{\xi_0 = -T, \quad \xi_1, \dots, \xi_n = T\}$$

is a division of R into n subintervals of positive length and

$$\max_{1 \le i \le n}|\xi_i - \xi_{i-1}| \to 0 \qquad \text{as } n \to \infty.$$

Using (13) with (14) for $l = 2$, we get

$$\mu_n(d\varphi) = C_n \exp\{-\tfrac{1}{2}[P_n\varphi, P_n\varphi]\} \prod_{i=1}^{n} d\varphi_i$$

which defines a weak distribution as $n \to \infty$. Since we need only ratios of integrals over Ω, the normalization constant C_n is not necessary and we can express $\mu_n(d\varphi)$ and $A[\varphi(\cdot)]$ and $B[\varphi(\cdot)]$ in terms of $\varphi = \Phi y$ and write

$$\int_{\Omega} \mu(d\varphi)\, F(A[\varphi], B[\varphi]) = C \int_{\Phi^{-1}\Omega} \mu(dy)\, F(A[y], B[y])$$

The constant C does not depend on y because Φ is linear and we do not need C for the same reasons as C_n. Finally $\mu_n(dy)$ is, by (14),

$$\mu_n(dy) = C_n \exp\{-\tfrac{1}{2}(P_n y, P_n y)\} \prod_{i=1}^{n} dy_i$$

The sequence $\{A_n\}_{n=1}^{\infty}$ of phase functions defining A in (5) is

$$A_n[\varphi] = \frac{n}{2}\int_{x+1/n}^{x-1/n} d\xi\, \varphi(\xi) = \int_{-T}^{T} d\xi\, a(\xi, x)y(\xi)$$

where

$$a_n(\xi, x) = \frac{x\xi}{2T} - \frac{T}{2} + \begin{cases} \frac{1}{2}|x-\xi|, & |x-\xi| > 1/n \\ 1/4n + 1/4n(x-\xi)^2, & |x-\xi| \leqslant 1/n \end{cases}$$

The Liouville operator $i\mathscr{L}$ is used in the form

$$i\mathscr{L}\cdot = \int_R d\xi\, Q(\varphi(\xi)) \frac{\delta}{\delta\varphi(\xi)\, d\xi} \tag{17}$$

which is valid only for $\mathrm{Re} \to \infty$. The $Q(\varphi)$ are given by

$$\dot{\varphi} = Q(\varphi) = -\varphi\frac{\partial\varphi}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2\varphi}{\partial x^2} - \varphi\frac{\partial\bar{u}}{\partial x} - \bar{u}\frac{\partial\varphi}{\partial x} + f(\bar{u}) \tag{18}$$

where

$$f(\bar{u}) = -\frac{\partial\bar{u}}{\partial t} - \bar{u}\frac{\partial\bar{u}}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2\bar{u}}{\partial x^2} \tag{19}$$

as a consequence of the splitting of the variable ν satisfying (11a) into

$$\nu(x, t) = \bar{u}(x, t) + \varphi(x, t)$$

The set of equations to be solved simultaneously is then

$$\frac{\partial\bar{u}}{\partial t} = -\frac{1}{2}\frac{\partial\bar{u}^2}{\partial x} - \frac{1}{2}\frac{\partial\overline{\varphi^2}}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2\bar{u}}{\partial x^2} \tag{20}$$

and Eq. (10) or (11), respectively, for $g(u, t)$ at each $x \in R$. By definition it follows that

$$\overline{\varphi^2}(x, t) = \int_\Omega dF\,[\varphi(\cdot), t]\varphi^2(x)$$

and from (5) it follows that for x fixed, $\varphi(x)$ can be identified with u or u', respectively. We have

$$\overline{\varphi^2}(x, t) = \overline{u^2} = \int_{-\infty}^{\infty} du\, g(u, x, t)u^2$$

Since g depends on x, the derivatives of $\overline{u^2}$ with respect to x exist and do not vanish identically. Furthermore (10) can be written in the form

$$\frac{\partial g}{\partial t} + W\cdot v\cdot\frac{\partial}{\partial u}\left(\frac{g}{W}\right) = \int_0^t ds \int_{-\infty}^{\infty} du'\, \frac{g(u', t-s)}{W(u')}\tilde{K}(u, u', s) \tag{21}$$

for initial conditions $f_2 = 0$. The coefficients are defined by (7) and

$$v(u, x) = [1/W(u)] \int_\Omega \mu(d\varphi)\, \delta(A[\varphi] - u) i\mathscr{L} A[\varphi] \tag{22}$$

$$\tilde{K}(u, u', s) = \int_\Omega \mu(d\varphi)\, \delta(A - u) i\mathscr{L} e^{-s(I-P_A)i\mathscr{L}} (I - P_A) i\mathscr{L}\, \delta(A - u') \tag{23}$$

5.1. Calculation of the Structure Function $W(u)$

To perform the calculation we make use of Ref. 14, from which it immediately follows that (Fig. 1)

$$W(u, x) = \frac{1}{(2\pi X)^{1/2}} \exp\left(-\frac{u^2}{2X}\right) \tag{24}$$

and

$$X = \lim_{n\to\infty} X_n, \qquad X_n = X(A_n)$$

$$X_n(x) = \int_R d\xi\, a_n^2(\xi, x) = (1/6T)(T^{-2} - x^2)^2 + O(1/n) \tag{25}$$

5.2. Calculation of the "Convective" Term $v(u, x)$

First we apply $i\mathscr{L}$ on A_n and get

$$i\mathscr{L} A_n[\varphi(\cdot)] = \tfrac{1}{2} n \int_{x-1/n}^{x+1/n} d\xi\, Q(\xi) \equiv \int_{-T}^{T} d\xi\, s_x(\xi) Q(\xi)$$

Inserting expression (18) into the integral and using the representation (15), we get after some lengthy calculations, according to Ref. 14,

$$v(u, x) = v_0(x) + f(\bar{u}) + u[(1/\text{Re}) v_1(x) + v_2(x)\bar{u}(x) - \tfrac{1}{2}(\partial\bar{u}/\partial x)] + 2u^2 v_2(x) \tag{26}$$

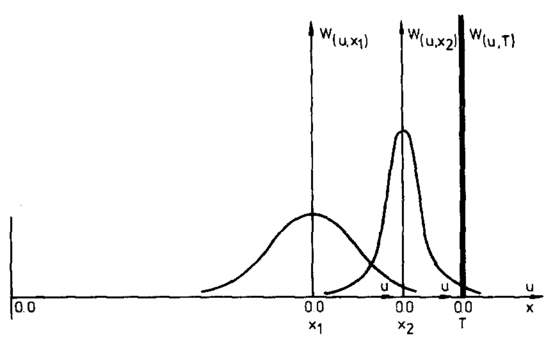

Fig. 1. Structure function $W(u, x)$ near fixed boundary (Burgers equation).

where

$$\begin{aligned} v_0(x) &= (x/3T)(x^2 - \tfrac{5}{2}T^2) + O(1/n) \\ v_1(x) &= -[3/(T^2 - x^2)] + O(1/n) \\ v_2(x) &= [x/(T^2 - x^2)] + O(1/n) \end{aligned} \tag{27}$$

and $f(\bar{u})$ is given by (19) (Fig. 2).

5.3. Calculation of the "Kernel" Function $\tilde{K}$

So far it has proven impossible to derive the result of the integration in (23) in closed form. The only way out is to calculate the terms in the (time) series expansion of the exponential operator $\exp\{-s(I - P_A)i\mathscr{L}\}$, which is extremely tedious and unsatisfactory because for the integration over Ω with respect to a weak distribution we have to prove in addition to convergence that integral and convergent series may be interchanged. But we can show at least that K is not identically zero by applying the projection operator P_A to $i\mathscr{L}A_n$. It is sufficient to consider the linear part of $i\mathscr{L}A_n$ given by

$$B_n[y(\cdot)] = \int_R d\xi\, s_x(\xi) y(\xi)$$

Then

$$P_A B_n = \frac{1}{W(A_n[y])} \int_\Omega \mu(dy')\, \delta(A_n[y'] - A_n[y]) B_n[y']$$

and from Ref. 14 it follows that

$$P_A B_n = \frac{Y_n}{2X_n} A_n[y], \qquad \frac{Y_n}{2X_n} = -\frac{3}{T^2 - x^2} + O\left(\frac{1}{n}\right)$$

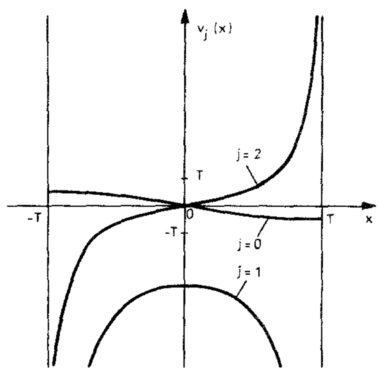

Fig. 2. "Convective" coefficients (Burgers equation).

and therefore

$$(I - P_A)B_n[y] = B_n[y] + \frac{3}{T^2 - x^2} A_n[y] + O\left(\frac{1}{n}\right)$$

which is obviously not identically zero. Therefore the zeroth-order term of $\tilde{K}$ is not identically zero and this implies that $\tilde{K} \not\equiv 0$ almost everywhere in R.

6. CONCLUSIONS

For the application of projection operator methods to turbulence the necessary tool of functional integration has been described and for the example of the Burgers model equation some integrals have been explicitly calculated. The proper construction of the integration process over the phase space Ω encounters some unexpected difficulties because of the nonexistence of translational-invariant integrals and because of the possibility that to a sequence of compatible finite-dimensional distributions there does not correspond a measure unless an additional condition [13] is satisfied by the sequence. The result of the formalism described above is a system of integrodifferential equations consisting of (a) equations for mean quantities containing second-order moments of the fluctuating component with spatial coordinates $\mathbf{x}$ and time t as independent variables; these equations are nonclosed in the classical sense; and (b) equations for the pdf of the values of an appropriately chosen phase function; here the fluctuating component at a point $\mathbf{x}$ is the independent variable; the coefficients (and therefore the solution) depend on the spatial coordinates $\mathbf{x}$ parametrically; this equation has to be solved at each $\mathbf{x} \in R$ theoretically. The connection between (a) and (b) is established by the definition of moments.

To illustrate the integration procedure the structure function $W(u)$ and the "convective" term $v(u, x)$ in the so-called master equation derived in Ref. 11 were calculated. The domain of all realizations was assumed to be a bounded interval with homogeneous boundary conditions for $\varphi \in \Omega$. The special form of the master equation (21) in this example suggests that for $t \to \infty$ and points $\mathbf{x}$ approaching the boundary from inside, the function $W(u, x)$ plays the role of an asymptotic solution. The result (24) for $W(u)$ shows that this is indeed the case because for $|x| \to T$ the function $W(u) \to \delta(u)$, as it should because we prescribed homogeneous and therefore deterministic boundary conditions. The results for $v(u)$ in (26) and (27) show some interesting features. The constant part (with respect to u) $v_0 + f(\bar{u})$ is uniformly bounded in R, whereas $|v_1|$ and $|v_2|$ go to infinity as x approaches either boundary. The origin of terms linear in u is easy to detect from (18) and the quadratic term in u and v_0 come from from the "inertial" term $\varphi(\partial\varphi/\partial x)$ in (18).

Finally the time correlation function, which has actually the character of a complicated differential operator, proved to be a stumbling block, but calculations based on series expansion in time and a Gaussian measure are under way for the Burgers model and shall be presented in a subsequent paper.

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REFERENCES

1. H. W. Wyld, Jr., *Ann. Phys.* **14**:143 (1961).
2. R. H. Kraichnan, *Phys. Rev.* **109**:1407 (1958); *J. Fluid Mech.* **5**:497 (1959).
3. R. Pythian, *J. Phys. A* 2:181 (1969).
4. R. Balescu and A. Senatorski, *Ann. Phys.* **58**:587 (1970).
5. J. R. Herring, *Phys. Fluids* **9**:2106 (1966).
6. S. F. Edwards, *J. Fluid Mech.* **18**:239 (1964).
7. J. Lee, *J. Math. Phys.* **15**:1571 (1974).
8. I. Prigogine and P. Resibois, *Physica* **27**:629 (1961).
9. E. Hopf, *J. Math. Mech.* **1**:87 (1952).
10. J. U. Keller, unpublished report, Aachen (1972).
11. R. Zwanzig, *Phys. Rev.* **124**:983 (1961).
12. S. L. Sobolev, *Einige Anwendungen der Funktionalanalysis,* Akademie Verlag, Berlin (1964).
13. A. V. Skorohod, *Integration in Hilbert Space*, Springer (1974).
14. W. Kollmann, *J. Math. Phys.* **16**:2195 (1975).